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THE LOGIC OF RUSSELL'S PARADOX AND THE SYNTHESIS OF TYPE THEORY

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Abstract
This paper explores the logical and philosophical meaning of the Russell paradox, and follows the intellectual path according to which the Russell paradox led Russell to one of the most impactful advances of the foundations of logic and mathematics: the theory of types. This paper rebuilt the original finding of Russell of the paradox in 1902, its destructive influence on the logicist programme of Frege, and the subsequent efforts to overcome it which have led to the ramified theory of types published in Principia Mathematica (1910-13). This article then proceeds to develop the history of type theory beginning with the simplification of type theory by Ramsey and its formalisation by Church, the Curry-Howard correspondence, the intuitionistic type theory based on Martin-Lof, the Calculus of Constructions and most recently, homo-type theory and the development of univalent foundations. In this article, it is argued that the paradox Russell had found was not a wholesome and irresolvable anomaly but a prodigal crisis that forced the re-examination of the nature of predication, collection and logical stratification by logicians and mathematicians, leading to a family of type-theoretic systems on which modern proof assistants, programming language theory and the future re-conceptualisation of the foundations of mathematics is now based.

Keywords: Russell, paradox, type theory, Principia Mathematica,

1. Introduction

During the summer of 1902, the fledgling philosopher of British common sense, Bertrand Russell, sent got his great German teacher of logic, Gottlob Frege a letter that was to change not only his own life, and that of his closest associates, but also to have an extended and far-reaching impact upon the development of thought and doctrine into the present century..[1] What was announced in the letter was a discovery of appalling naivete in effusion of a shocking bearing; that the naive notion of set or class, which allowed any given property to define a set of all and only those objects which had that property, was incompatible. When Russell applied the idea of not being a member of itself to classes, it brought up a contradiction in that property: the class of all classes that are not a member of themselves is a member of itself only in case it is not a member of itself. This led to the discovery of what would later be popularly referred to as Russell paradox that highlighted a major weakness of the logical foundations upon which Frege had built his ambitious logicist programme of reducing arithmetic to pure logic..[2]

This paradox has forced a wholesale re-evaluation of the principles of predication, the formation of classes and logical generality. The systematicity of the responses to this crisis was most evident in that of Russell himself, the theory of types, which held the universe of logical and mathematical objects stratified into a hierarchy of levels, and the paradoxical constructions producing contradictions were ruled out by limiting the admissible combinations of entities in levels. The theory of types, first outlined in a supplement to The Principles of Mathematics (1903) by Russell, expanded in his 1908 paper Mathematical Logic as Based on the Theory of Types, and realised in the monumental Principia Mathematica (1910-13) he co-worked on with Alfred North Whitehead, became one of the classical forces behind logic during the twentieth century..[345]

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[1]Bertrand Russell, Letter to Frege, 16 June 1902, reprinted in Jean van Heijenoort, ed., From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931 (Cambridge, MA: Harvard University Press, 1967), 124–25.

[2]Gottlob Frege, Grundgesetze der Arithmetik, vol. 2 (Jena: Hermann Pohle, 1903), appendix, 253–65.

[3]Bertrand Russell, The Principles of Mathematics (Cambridge: Cambridge University Press, 1903), 78–105

[4]Bertrand Russell, The Principles of Mathematics (Cambridge: Cambridge University Press, 1903), 100–104

[5]Georg Cantor, "Über eine elementare Frage der Mannigfaltigkeitslehre," Jahresbericht der Deutschen Mathematiker-Vereinigung 1 (1891): 75–78.

This paper will take an in-depth analysis of the reasoning behind Russell paradox and the synthesis of type theory using exposition and critical analysis of literature. The expository aspect re-creates the paradox in the context of history and logic, and follows backward the preconditions of the diagonal argument of Cantor as well as the Burali-Forti paradox of the largest ordinal. The analytical dimension works in the rich two-level literature about the paradox and its solution, critiquing the theory of Ramsey, the formulation theory of Church and the radical revitalization of the theory by Curry, Howard, Martin-Lof, Coquand and the architects of the homotopy type theory. The notes-bibliography system of the seventeenth edition of The Chicago Manual of Style is the followed one by the documentary apparatus..

2. Logical Structure of Russell's Paradox

Russell comes up with his paradox on the basis of the unrestricted principle of comprehension according to which on the one hand, given any property or predicate P, we can form a set S, such that on the one hand, S has all objects, on the other hand, all objects are in S. This was implicit in the work of Cantor, explicit in Axiom V (Basic Law V) of the Grundgesetze der Arithmetik of Frege, and common intuitive knowledge among the set-theoretic work of the nineteenth-century mathematicians. The most devastating insight of Russell was to treat one of the specific predicates, which was not a member of itself, is not, and to invoke the principle of comprehension on this predicate. Let R be the collection of all collections which are not subdivisions of themselves: $R = \{x:x \notin x)$. Then the question comes as to whether or not R is a member of R. Since $R \in R$, according to the defining property of R, $R \notin R$--a contradiction. The fact that $R \notin R$ implies that R has the defining property of membership in R hence $R \in R$ again a contradiction..

The paradox was not born out of nothing. It can be logically analysed as closely allied to the diagonal argument of 1891 of Cantor which had shown that the power set of any set is stricter than the set which it contains by constructing, given any proposed bijective, a set which the bijective would necessarily fail to contain.[6] That the naive conception of totality is full of danger had already been indicated by the Burali-Forti 1897 paradox demonstrating that the assumption of a greatest ordinal number is inconsistent.[7] Russell himself saw the resemblance between his paradox and the earlier results and in The Principles of Mathematics he studied them as examples of a general logical form: the effort to construct a totality which, by its very definition, must contain itself and exclude itself..[8]

Michael Dummett has stated that the importance of Russell paradox is not only that of its technical subject matter but that it demonstrates a very basic philosophical question regarding the essence of predication and collection.[9] The paradox shows that not all grammatically correct predicates define a set, and the connexion between the intension of a concept (the defining property of it) and the extension (the set of objects that satisfy this definition) is not as immediate as the naive comprehension principle would allow it to be. This intuition would become the impetus behind the development of theory of types, axiomatic set theory and practically all other attempts to study foundations of mathematics.

3. Russell's Initial Responses and the Genesis of Type Theory

The earliest effort by Russell to deal with the paradox was in an appendix to The Principles of Mathematics (1903), in which Russell had outlined an early form of what he termed the doctrine of types.[10] The fundamental one was that objects of the logical universe are organised into a hierarchy of types: at the bottom end we have individuals (objects that are not sets); in the next end we have sets of individuals; in the next end sets of sets of individuals, and so forth. The most important limitation is that the entities of a given type can only belong to a set of the type right beneath it. Operating within this inhibition, the so-called paradoxical set R R -: the set of those absolutely or unconditionally not (belonging to) some set, cannot be constituted, in that performing the operation to consider whether a set is a object of set involves in the application of a set to a type of an argument, which the type discipline prohibits. The stratification of logical universe into levels destroys the self-referential loop which creates the paradox.

The complete theory of types was introduced by Russell in his 1908 work Mathew Logic as Based on the Theory of Types which introduced the theory of ramified types. The ramified theory does not only distinguish between the types of entities but also the orders of propositions within the type of entities. A proposal which quantifies over entities and propositions of order, in fact, less than n is called a proposal of order n. The extra stratification was encouraged by Russell studying the semantic paradoxes, the Liar paradox and its variants, where Russell studied self-referential propositions instead of

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6Cesare Burali-Forti, "Una questione sui numeri transfiniti," Rendiconti del Circolo Matematico di Palermo 11 (1897): 154–64.

7Bertrand Russell, The Principles of Mathematics (Cambridge: Cambridge University Press, 1903), 497–528.

8Bertrand Russell, "Mathematical Logic as Based on the Theory of Types," American Journal of Mathematics 30, no. 3 (1908): 222–62.

9Bertrand Russell, "Mathematical Logic as Based on the Theory of Types," American Journal of Mathematics 30, no. 3 (1908): 224–30.

10Alfred North Whitehead and Bertrand Russell, Principia Mathematica, 3 vols. (Cambridge: Cambridge University Press, 1910–13), 1:37–65.

self-referential sets. Russell has found these paradoxes to be caused by a vicious circle: the effort to define a matter by the means of totality to which the very object is a member. The ramified theory of types was to impose the vicious circle principle, demanding that all definitions be predicative, i.e. referring only to totalities that can be built without the defined entity.[11]

More than this philosophical and technical work has been detailed by Ivor Grattan-Guinness, which Russell had invested in developing the ramified theory, and Russell had wrestled with the problem of obtaining a satisfactory solution to the paradoxes before achieving the system that he came up with in 1908.[12] The multiple midway suggestions which Russell took and rejected such as zigzag theory, the limitation-of-size theory, and the no-class theory are evidence of the complexity of the issue and the cleverness with which Russell tackled it.[13]

4. Principia Mathematica and the Axiom of Reducibility

Principia Mathematica, a three-volume treatment of logic by Alfred North Whitehead and Bertrand Russell, published between 1910 and 1913, is by far the most ambitious project in the history of logic to deduce the entirety of pure mathematics out of a few logical axioms. The logical system of the Principia is the ramified theory of types, with a few axioms of less or more disputable logical status. Among these, the most important one is the axiom of reducibility, that there is to each propositional functional, of every order, a coextensive predicative functional of the lowest order. To offset these expressive limitations that the ramification of types imposed on the system, the axiom of reducibility was added: without the axiom of reducibility, several well-known constructions of mathematics objects, such as the least upper bound principle of real numbers and the principle of mathematical induction in its full generality, could not be constructed within the ramified hierarchy.[14]

The axiom of reducibility was instantly known to be problematic. It seemed an assumption on a substantial mathematical level, not a logical truth, and its presence in the system had the effect of weakening the logicist programme of reducing mathematics to logic. In his masterpiece essay upon the mathematical logic of Russell, Kurt Gödel wrote that the axiom of reducibility effectively collapses the ramified hierarchy, and the difference between the orders otiose and the reducing of the ramified theory to a simple theory of types where only the difference of types (and never the difference of orders) is effective.[15][16] According to Grattan-Guinness, the axiom of reducibility has been a point of long-standing dissatisfaction to Russell himself, who admitted that it was not a self-evident propositional truth as a proper logical axiom needed to be..

The Principia Mathematica philosophical legacy is complicated. On the one hand it showed (with a kind of formal rigour never previously attained) that much of classical mathematics was reducible to a type-theoretic logical system, and was thus able to justify at least the spirit of the logicist programme. Conversely, the dependence on the axiom of reducibility, the axiom of infinity (that says that there are infinitely many individuals), and the axiom of choice posed searching questions about the limits between logic and mathematics which have never been entirely answered. Philippe de Rouilhan has maintained that the Principia lies at an intermediate stage in the chronicle of logic: one on the one hand, the culmination of the logicist tradition, and on the other, the work which most thoroughly exposed the shortcomings of the tradition..[17]

5. Ramsey's Simplification and the Simple Theory of Types

In 1926, the Cambridge mathematician Frank Ramsey made the first significant revision of the type theory of Russell in his article The Foundations of Mathematics.[18] The key input made by Ramsey was his reflection that the issues of inconsistencies that the ramified theory of types was supposed to avoid are divided into two essentially different categories. The first category is the logical or set-theoretic paradoxes e.g., the paradox of Russell and the Burali-Forti

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[11] Alfred North Whitehead and Bertrand Russell, Principia Mathematica, 3 vols. (Cambridge: Cambridge University Press, 1910–13), 1: 54–60. .

[12] Frank P. Ramsey, "The Foundations of Mathematics," Proceedings of the London Mathematical Society, ser. 2, 25 (1926): 338–84.

[13] Frank P. Ramsey, "The Foundations of Mathematics," Proceedings of the London Mathematical Society, ser. 2, 25 (1926): 350–60.

[14] Alonzo Church, "A Formulation of the Simple Theory of Types," Journal of Symbolic Logic 5, no. 2 (1940): 56–68.

[15] Alonzo Church, "A Formulation of the Simple Theory of Types," Journal of Symbolic Logic 5, no. 2 (1940): 58–62.

[16] Per Martin-Löf, "An Intuitionistic Theory of Types: Predicative Part," in Logic Colloquium '73, ed. H. E. Rose and J. C. Shepherdson (Amsterdam: North-Holland, 1975), 73–118.

[17] Per Martin-Löf, "An Intuitionistic Theory of Types: Predicative Part," in Logic Colloquium '73, ed. H. E. Rose and J. C. Shepherdson (Amsterdam: North-Holland, 1975), 80–95.

[18] Per Martin-Löf, Intuitionistic Type Theory: Notes by Giovanni Sambin of a Series of Lectures Given in Padua, June 1980 (Naples: Bibliopolis, 1984), 1–30.

paradox which entail the use of self-referential constructions in the context of sets and classes. The second subdivision includes the semantic or epistemological paradoxes, which include such as the Liar paradox and Berrys paradox, a paradox involving self-referential constructions on the field of language, truth and the definability. According to Ramsey, the ramification of the type hierarchy, which had been assumed necessary to resolve the semantic paradoxes, was not necessary to the resolution of the logical paradoxes, and the semantic paradoxes needed to be solved in some other way, namely, that a precise separation was made between the logical and the linguistic.[19]

The endeavour to eliminate the ramification and reduce simple type differences to simple ones brought Ramsey to the simple theory of types, in which types classify entities, propositions and propositional functions are not stratified by order any further. The theory of types with its simple form turned out to be much easier to handle than the ramified theory, and it could be used in order to prove the necessary mathematical constructs without the axioma of reducibility. Alonzo Church adopted and provided a formal based simplification of this simplification in his historic 1940 paper A Formulation of the Simple Theory of Types.[20] The formulation of Church has married the simple type hierarchy with the lambda calculus, which he coined by himself, to result in a system of unparalleled elegance and power. In the formulation of Church, any constituted expression is attributed to a type and the discipline of the type guarantees that one is not able to construct self referentially the type of construction, the construction of the paradox. The lambda calculus offers a consistent language in defining functions and the typing system restricts the usage of functions to inputs in a manner that ensures consistency..[21]

6. The Curry–Howard Correspondence and the Computational Interpretation of Types

Among the most significant and far-reaching achievements in the history of type theory is the fact that logical proofs (and computational programmes) significantly share powerful structural relationships that are called the CurryHoward correspondence or propositions-as-types paradigm. It was expected in the 1930s and 1940s in the work of Haskell Curry and formulated by William Howard in a manuscript circulated in 1969 and published in 1980..[2223] The Curry–Howard correspondence establishes that under a suitable interpretation, the types of a typed lambda calculus correspond to the propositions of intuitionistic logic, the terms of each type correspond to proofs of the corresponding proposition, and the rules of computation (beta-reduction) correspond to the normalization of proofs. In this framework, the type A → B corresponds to the logical implication "if A then B"; a function of type A → B is a proof that transforms a proof of A into a proof of B; and the product type A × B corresponds to the conjunction "A and B."[24]

According to the CurryHoward correspondence, type theory and proof theory are more than corresponding theories: they are structurally identical: any type system is also a logical system, and indeed any logical system (of the right kind) is also a type system. This recognition has been far-reaching to both logic and computer science. It has also enjoyed theoretical foundations in the design of the programming languages which are dynamically typed and which have a type system which is an automatic proof-checking system that ensures that some properties of the programmes will be true at compile-time. The evolution of programming languages, both inspired by type theory (ML and Haskell) and by its CurryHoward correspondence (Scala, Rust, and the dependently typed programming languages underlying current proof assistants), has been described by Benjamin Pierce..[2526]

7. Martin-Löf's Intuitionistic Type Theory

The greatest, and philosophically deepest, extension of the Curry-Howard correspondence would be the intuitionistic type theory of the Swedish logician and philosopher Per Martin-Löf, originally in 1973 but later developed during the next

[19] Thierry Coquand and Gérard Huet, "The Calculus of Constructions," Information and Computation 76, nos. 2–3 (1988): 95–120.

[20] Thierry Coquand and Gérard Huet, "The Calculus of Constructions," Information and Computation 76, nos. 2–3 (1988): 100–110.

[21] The Univalent Foundations Program, Homotopy Type Theory: Univalent Foundations of Mathematics (Princeton: Institute for Advanced Study, 2013), 1–30.

[22] The Univalent Foundations Program, Homotopy Type Theory: Univalent Foundations of Mathematics (Princeton: Institute for Advanced Study, 2013), 75–120.

[23] Vladimir Voevodsky, "The Origins and Motivations of Univalent Foundations," The Institute Letter, Summer 2014, 8–9.

[24] Michael Dummett, Frege: Philosophy of Mathematics (London: Duckworth, 1991), 211–30.

[25] W. V. Quine, "New Foundations for Mathematical Logic," American Mathematical Monthly 44, no. 2 (1937): 70–80.

[26] Ernst Zermelo, "Untersuchungen über die Grundlagen der Mengenlehre I," Mathematische Annalen 65 (1908): 261–81.

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several decades.[27] The type theory of Martin-Lof is both a theory of constructive mathematics and a proof calculus as well as a programming language. It is a generalisation of the simple typed lambda calculus, including a dependent type of types that can depend on the values, and thus attains a level of expressiveness that makes it by far out do the simple theory of types. In the system of Martin-Löf, arbitrary logical properties of an object can be coded by the type of the object, in such a way that the construction of a term of one type also makes clear the object satisfies the property coded by the type..[28]
The theoretical importance of the type theory created by Martin-Ldoff is that it is constructive, and addresses the problem of existence and construction. In classical logic, to show that an object, with a specified property, exists, one merely has to construct a contradiction between the assumption that such an object exists and the fact that it does not exist. Martin-Löf also defines intuitionistic frameworks in which the proof of existence should literally provide a witness, a real object as well as a demonstration that it has the necessary property. This positive sense of logic and mathematics, intermediated by the system of type-theories, is the direct heir of the tradition begun by Brouwer and Heyting but is one that has a degree of formal precision and of computational content not previously possessed by the intuitionistic tradition..[29]
The type theory of Martin-Löf, has turned out to be remarkably fruitful both as regards its philosophical and as regards its computational aspects. Philosophically, it offers a rigorous system of the constructive underpinnings of mathematics where the concepts of proof, construction as well as computation are harmonised. It has been used on the computational side, as the theoretical foundation of a family of proof assistants, including Agda, directly built upon the system of Martin-Lof, and has been used to provide formal verification of mathematical proofs and computer programmes. Calculus of Constructions, designed in 1988 by Thierry Coquand and Gerard Huet, is a stronger system incorporating dependent types as well as polymorphism in a system of exceptional generality, and provides the logical basis of the Coq proof assistant, which is among the most popular formal verification systems in mathematics and computer science..[30][31]

8. Homotopy Type Theory and Univalent Foundations

The last and arguably the most radical advance in the type theory process is the development of homotopy type theory and the univalent foundations programme, first started by Fields Medalist Vladimir Voevodsky, and further developed through a large-scale network of contributors to the publication of the book Homotopy Type Theory in 2013..[32] The development of homotopy type theory brings about the amazing realisation that the types and the identity types of Martin-Lof intuitionistic type theory can be interpreted naturalistically in terms of homotopy theory, which is a subdivision of algebraic topology that studies the properties of spaces that are continuous to be deformed. Types are identified by this definition as spaces, terms of a type are objects of a space, and identity prove are those which establish a path between objects. Stronger identity evidences - evidences that two identity evidences are likewise of the same nature - are related to homotopies of paths, and the whole structure of homotopy theory of higher rank is embodied in the identity types of the type theory.[33]
The key, and defining, innovation of univalent foundations is such axioms as the univalence axiom, introduced by Voevodsky, that proposes that types are identical in case they are equivalent.[34] In the homotopical interpretation, this axiom is an expression of the fact that homotopy equivalent spaces are indistinguishable in the type theory- a fact which reflects a more profound and pervasive phenomenon in mathematical practise, in which the treatment of indistinguishable structures that are homotopy equivalents as being identical is a routine practise. The univalence axiom offers a formal defence of this practice which we cannot give in a traditional set-theoretic foundation, where equivalent non-identical structures have to be bifferently discriminated.

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[27]Gregory H. Moore, Zermelo's Axiom of Choice: Its Origins, Development, and Influence (New York: Springer, 1982), 1–45.

[28]Ivor Grattan-Guinness, The Search for Mathematical Roots, 1870–1940 (Princeton: Princeton University Press, 2000), 310–55.

[29]Ivor Grattan-Guinness, The Search for Mathematical Roots, 1870–1940 (Princeton: Princeton University Press, 2000), 350–80.

[30]Bertrand Russell, "Mathematical Logic as Based on the Theory of Types," American Journal of Mathematics 30, no. 3 (1908): 240–50.

[31]Alfred North Whitehead and Bertrand Russell, Principia Mathematica, 3 vols. (Cambridge: Cambridge University Press, 1910–13), 1:165–90.

[32]Kurt Gödel, "Russell's Mathematical Logic," in The Philosophy of Bertrand Russell, ed. Paul Arthur Schilpp (Evanston, IL: Northwestern University Press, 1944), 123–53.

[33]Kurt Gödel, "Russell's Mathematical Logic," in The Philosophy of Bertrand Russell, ed. Paul Arthur Schilpp (Evanston, IL: Northwestern University Press, 1944), 135–45.

[34]Philippe de Rouilhan, Russell's Logicist Definitions of Numbers (Paris: Éditions de l'École Normale Supérieure, 2015), 1–40.

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Homotopy type theory has far-reaching implications on the foundations of mathematics, which are being studied. The programme of univalent foundations suggests to base math on other foundations other than set theory, namely a type-theoretic system with the unitary objects not a set, but a type with a detailed higher-dimensional geometry. This proposal has created sharp interest and discussion by the mathematical community. Advocates of univalent foundations posit that they give a more natural and computationally manageable foundation to mathematics, and is closer to how mathematics is actually practised in the real-world, and makes complex mathematical arguments verifiable using proof assistants..[35]
The practise of the implementation of foundations of large-scale formalised mathematics on a type-theoretic basis has shown itself viable in practise by the formal verification of significant mathematical conjectures, including the FeitThompson odd order theorem, which was certified in the Coq proof assistant through its leadership by Georges Gonthier.[36]

9. Type Theory, Programming Languages, and Verified Software

This impact of type theory is much broader than the principles of mathematics, and into the real world of software engineering and programming language design. All modern statical typed programming languages have a substantial debt to the type-theoretic tradition of Russell starting with his reaction to his own paradox. The type systems of languages that include ML, Haskell, OCaml, Scala, Rust, TypeScript, and Swift are lineal, by multiple developmental paths, descendants of the simple theory of types as formalised by Church and the CurryHoward correspondence illustrated it. Pierce has claimed that type theory is the source of the conceptual vocabulary, and formal means, of the design of programming languages where the entire categories of errors, type mismatches, null pointer dereferences and some concurrency violations, are avoided at a compile time by the type system the discipline..[37]
The relationship between type theory and software verification has gained meaning in a world where the accuracy and reliability of software systems have engulfing implications on human lives and the welfare of a society. The CompCert project pioneered by Xavier Leroy established that one can construct a fully-validated optimising compiler to the C programming language, using only Coq hint and the type-theoretic foundations it offers.[38] The checked compiler provides a mathematical verification that the code that is compiled accurately reflects the behaviour that was specified in the source programme- an assurance that cannot be achieved by any conventional software testing approach. The increased popularity of the use of proof assistants and dependent type theories in industry software development, whether in the form of verified operating system kernels or formally verified cryptographic protocols, is evidence of the practical significance of the type-theoretic tradition which Russell had pioneered.

10. Alternative Responses: Axiomatic Set Theory and the Wider Legacy

It should be pointed out that not the only answer to the paradox of Russell was the theory of type. The strategy that gradually became dominant in the 20th century of the mainstream of mathematics was axiomatic set theory, specifically the system of Ernst Zermelo, 1908, and later enlarged by others to what is now Zermelo-Fraenkel set theory including the axiom of choice (ZFC).[39] Zermelo tried to follow this principle by a limiting principle of comprehension, called the axioms, which allowed the sets to be constructed out of already known sets only, avoiding the constructions of the paradox which is self referential. Gregory Moore has written the involved historical relations between the axiomatization of Zermelo, type theory of Russell and the community of mathematicians in general seeking answers to foundational issues during the first half of the twentieth century..[40]
W. V. Another point of view was the system of New Foundations that was proposed by Quine and that contains a variation of the comprehension principle but it is limited by a syntactic stratification condition modelled after type theory. The multiplicity of reactions to Russell paradox Russell discovered a rich problem indeed, and the responses to it defy classification, by type theory, by axiomatic set theory, by stratified comprehension, by constructive foundations. All of the responses reflect a philosophical position on the question of what mathematical objects are, the extent of logical principles, and the connexion between logic and mathematics, and the debate on which the approaches remain continues to inform the history of mathematics in the twenty-first century..

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[35]Haskell B. Curry and Robert Feys, Combinatory Logic, vol. 1 (Amsterdam: North-Holland, 1958), 1–40.

[36]William A. Howard, "The Formulae-as-Types Notion of Construction," in To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, ed. J. P. Seldin and J. R. Hindley (London: Academic Press, 1980), 479–90.

[37]William A. Howard, "The Formulae-as-Types Notion of Construction," in To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, ed. J. P. Seldin and J. R. Hindley (London: Academic Press, 1980), 482–88.

[38]Benjamin C. Pierce, Types and Programming Languages (Cambridge, MA: MIT Press, 2002), 1–25.

[39]Benjamin C. Pierce, Types and Programming Languages (Cambridge, MA: MIT Press, 2002), 339–70

[40]Xavier Leroy, "Formal Verification of a Realistic Compiler," Communications of the ACM 52, no. 7 (2009): 107–15.

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11. Conclusion

One of the most significant events on the intellectual history of the twentieth and twenty-first centuries is the synthesis of the idea of type theory and the logic of Russell paradox. The observation made by Russell that the free law of comprehension results in inconsistency revealed the underlying instability of the premises of logic and mathematics and forced a radical reevaluation of the laws of predication, collection and logical generality. Russell's theory of types, the theory he came up with in reaction to this crisis, enshrined a principle of logical stratification, which has turned out to have long-term significance, both in the foundations of mathematics and in the structure of programming languages, in the practise of formal verification, and in the continued reconceptualization of the interrelations between logic, computation, and mathematical structure.[41][42]

The history of Russell's ramified theory being carried by Ramsey to his simplification and Church to his formalisation to Martin-Lof with his intuitionistic type theory, the Calculus of Constructions, and homotopy type theory, as an exemplification of this history, stands as an impressive urbanisation of ideas a process of synthesis, which started with the solution to a logical paradox. CurryHoward correspondence showed that the stratification of the logical universe explained by type theory was not just a bamboozle mechanism keeping the predicate logic paradox-free, but a structural phenomenon, connecting the deduction between propositions and the programmes, between propositions and types, between logical deduction and computational evaluation. A synthesis has been extended to an algebraic topological level by homotopy type theory and univalent foundations, who suggest a new vision of mathematical foundations that associates the working practises of mathematicians with a strongly structured space of types.

The paradox which Russell has described to Frege in a short note in 1902 was, however, not a disaster as it turned out, but a stimulus: a crisis which pushed the logical and mathematical community into the creation of new conceptual instruments of an unusual power and generality. The most persistent and fruitful heritage of that crisis, in its various forms and developments, is the theory of types, and its influence is still increasing with additional links between type theory, mathematics and calculation being unearthed and exploited. This is, in the most fundamental way, a narrative of the generative capacity of rigorous thought faced with its limitations; or, more precisely, a narrative of the generative capacity of the set of formal suspicions that constitute and mark that creative power.

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