APPRAISAL OF LOGICISM IN GOTTLOB FREGE

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ABSTRACT
This paper gives a critical evaluation of the logicist programme of Gottlob Frege where Frege insists that the truths of arithmetic can be reduced to, and derive out of, the laws of pure logic. This paper, via the exposition and critical analysis of the literary texts, recreates the philosophical inspirations, formal structure, and eventual destiny of the logicism of Frege as it was manifested in his three main publications: Begriffsschrift of 1879 (Frege's Concept Script), Die Grundlagen der arithmetik of 1884 (Foundation of Arithmetic) and Grundgestze der arithmetik of 1903 (Basic Laws of Arithmetic).This paper includes an analysis of how Russell paradox was devastating to the Frege system, it also evaluates the neo-Fregean revival that followed, involving Crispin Wright and Bob Hale, and also reviews the modern importance of the logicist ambitions of Frege due to the contributions of George Boolos, Richard Heck, John Burgess, and Øystein Linnebo. This paper claims, was the failure of the original logicist programme of Frege, not its strict form, which continues to offer some of the most fruitful and far-reaching conceptualizations of the nature of number, the analytic-syntactic distinction, the connexions between the field of mathemetics and the world of logic in philosophy.

Keywords: *Frege, logicism, arithmetic, Grundlagen, Grundgesetze, and neo-Fregeanism.*

INTRODUCTION

Philosophy of mathematics is faced with lots of questions such as; "are mathematical truths results of generalisations of empirical intuitions or the synthetic intuitions of a priori character or an analytic implication of logic"? Frege has dealt with this question more rigorously, and more ultimately than any other thinker in his

intellectual career, whose main method was to establish the truths of arithmetic on purely logical laws, hence the term logicism. The programme that Frege advocated was no simple technical task of formal derivation, it was a first-order argument, one that would show that knowledge of number typically based on the universal and necessary laws of thought itself.[1]

The current paper provides an extensive evaluation of logicism as practised by Frege by using exposition and literature criticism as its tools. The expository aspect recreates the philosophical inspirations, conceptual discoveries and formal equipment of the programme of Frege as expressed in his three major works. His first work, "Begriffsschrift" (Frege's Concept Script) of 1879 centers on his revolutionary logical notation. Frege's second writing, "Die Grundlagen der arithmetik" centred around the "Foundation of Arithmetic" of 1884. Here he demonstrated his Foundation of Arithmetic and the philosophical birthmark of his logicist thesis. His third work, "Grundgestze der arithmetik" of 1903 (Basic Laws of Arithmetic), Frege stated basic laws of arithmetic and tired proving truths from logical axioms[2]. This direction of analysis is the most critical approach to the vast body of secondary literature, assessed, defended, revised and contested by Frege logicism, namely, the writings of Michael Dummett, John Burgess, Øystein Linnebo, Joan Weiner, and Tyler Burge. The documentary apparatus is adhering to the notes-bibliography style of the seventeenth edition of The Chicago Manual of Style.

LOGICISM IN FREGE

Frege's logicist programme was greatly influenced by the late nineteenth century philosopher, Kant. In his very popular book, *Critique of Pure Reason,* Kant drew a strong synthesis on the philosophical war between the rationalist and empiricists such that arithmetic truths are seen as synthetic a priori judgement. This implies that such propositions as 7 + 5 = 12 is not analytic but synthetic a priori judgment. Thus, the truth of this proposition

cannot be obtained by simple examination of the concepts that it contains. Rather the truth of this proposition must be arrived at, by a synthesis that the faculty of pure intuition, in this case, achieves, that is the synthesis which some unity produces in us through the successive addition of units in the form of inner sense.[3] This Kantian doctrine was, philosophically speaking, a continual rejection of logicism as taken by Frege. Frege did not reject the Kantian classification of geometry as synthetic a priori, based on spatial intuition, but claimed that arithmetic was entirely different: its truths, he said, are analytic, and derive solely out of definitions and logical laws, and refer to intuition of any kind, either spatial or temporal.[4]

Frege was also against empiricist explanation of arithmetic as was connected with John Stuart Mill who had also argued that the truth of arithmetic is an inductive generalisation of sensory perception. Frege brought this to a disastrous conclusion in the Grundlagen where he noted that it causes confusion of how to apply number with the nature of number itself and makes arithmetical truths conditioned on experience in a manner that is incongruent with their apparent necessity and universality.[5] The psychologistic account, which Frege saw as common among some German logicians and mathematicians of his era, which identified logical and arithmetical truths with the psychological processes of their apprehension, was also rejected by Frege. Opposing this opinion, Frege clearly maintained a firm separation of subjective acts in thinking from the objective thought content, hence logical laws are not the description of how people tend to think, but prescriptions governing the relations between objective propositions.[6] Joan Weiner has claimed, convincingly, that the anti-psychologism of Frege was not itself only a polemical position but an underlying commitment, which defined the whole construction of his logical and philosophical work as creating the autonomy to logic as a pursuant of objective truth[7].

BEGRIFFSSCHRIFT AND THE INVENTION OF MODERN LOGIC

The most radical action in the logicist programme of Frege was the construction of a new logic language which he named the Begriffsschrift or concept-script, which was published in 1879.[8] The Begriffsschrift was a kind of formal predicate logic: a sort of formal language equipped with variables taking values over objects, quantifiers binding those variables, truth-functional connectives, a strictly defined system of inference rules, and a thoroughgoing ability to derive theorems out of axioms, such as had never been experienced in the Aristotelian syllogistic, or in the various early modern analogues of it. Thus, predicate logic is referred as first order logic as different from higher order logic. Higher order quantification refers to a formal system extending first order logic by allowing quantification over predicates or functions than individual variables. Thus, it can be said that the logical notation developed by Frege made distinction between predicate logic or the first order logic as against the higher order quantification. Thus, Frege by this means presents, a completely explicit system of formal deduction such that all the steps of any reasoning process is explained by an explicit rule.[9]

It is hard to overestimate the significance of the logicist programme to the Begriffsschrift. Prior to Frege, no logical language was powerful enough and precise enough to allow mathematical reasoning to be formalised. The conventional syllogistic was capable only of very limited types of inferences which concerned categorical theses, and the heterogeneous logics of Boole and his successors were more expressive but lacked the quantificational facilities required to codify the type of reasoning which pervades mathematics, reasoning about all numbers, about some functions, or about the existence of objects, which conform to certain conditions. The construction of the quantifier by Frege provides the structural means necessary to express and deduce arithmetic propositions upon purely logical foundations. Thus, like Dummett

asserted, "Begriffsschrift" in Frege has in the world of logic produced similar effect in which "Begriffsschrift" in Copernicus produced in astronomy, hence enlivening logic, previously a dead scholastic science, into a vibrant and lively science.[10]

"THE GRUNDLAGEN": PHILOSOPHICAL FOUNDATIONS OF LOGICISM

While Begriffsschrift furnished the formalism of the logicism of Frege, it was the Grundlagen der Arithmetik of 1884 that gave it the philosophical groundwork.[11] The Grundlagen is particularly concerned about the clear structure of argumentation, devastating criticism, and positive vision of philosophy. The work has two goals which are interdependent in that, in the first place, it aims to define what numbers are, and in the second place, it aims to be able to prove that the truths about numbers are analytic. This claim implies that the truths of numbers are of logical derivatives as defined in axioms of logic or laws of logic.

The work, "Grundlagen" begins and is anchored on a set of methodological principles. The paramount of these is what Dummett has referred to as the three principles of the Grundlagen namely the injunction to keep the psychological and the logical distinct, the admonition never to set in isolation the meaning of a word. Thus, meaning here ought not to be set in isolation rather borne from the proposition, hence referred as the context principle, and the need to distinguish a concept and object.[12] These postulates act as methodological scaffolding of the constructive argument which follows.

Having annihilated the competing accounts of the number by the empiricists, the psychologicists and the formalists, Frege proceeds to his own positive account. According to him, stating a number implies a clause which is better understood from the evolving context. For instance, a claim that Jupiter has four moons implies a

statement not about the moons, as physical objects themselves, but about the concept moon of Jupiter. This concept holds the number four as the object due to the number of objects that are subordinate to it. Thus, here number is mere property of another concept not of aggregates of material objects. This places the category of number squarely in the logical domain of concepts and concept extensions, as opposed to the domain of empirical objects, which are perceived by sense.[13]

The main question with which Frege faces the next challenge is the way numbers, as logical objects, can be defined. His response continues by what has now become known as the method of abstraction. By method of abstraction, Frege defines abstracts objects such as numbers by extensional and logical analysis and not by psychological or mental products. He employs Hume's Principle by identifying matching pairs between concepts. Thus, the definition of number of a concept A could be likened to the extension of the concept equinumerous to A. Similarly, if the number of the concept A is the same as the number of the concept B, hence one could say that objects in both correspond to each other in the ratio of one to one. This form of numerical identity and correspondence is similar to Hume's Principle who also claimed that the number of concept A and concept B can be identical if the number of things under the concept A and B has a one on one correspondence between them such that they can be described as bijection[14]

"THE GRUNDGESETZE": THE FORMAL DERIVATION AND ITS COLLAPSE

One of the most popular works of Frege is his Grundgesetze der arithmetik published in 1893 and 1903[15]. In this work, Frege shows a systematic construction of arithmetic theorems from expressly-stated logical axioms in the sophisticated formulation of Begriffsschrift. Frege's constructions on the Grundgesetze can be referred to as he Basic Law V. This law considers same value

functions, hence the extension of the function of A could be said to be the same regarding the extension of the function B in the case they both assign the same value to every argument[16]. Thus, Frege's Grundgesetze can be said to be centered around the theory of functions, concepts and truth-values. It refers to an axiom defining the relationship of extension of A with the extension of B.

In this work, Frege re-emphasized his earlier stand on the difference between sense (Sinn) and reference (Beteuntung) as already distinguished in his 1892 Easy on "Sense and Reference"[17]. Also, in "Function and Concept" of 1891, Frege presented a semantic systematic treatment of the concept as an object-to-truth-value.[18] In this context, any well-constructed expression of the Grundgesetze has a sense and has a reference, and the logic axioms govern the connexions between the references of the expressions, the objects, functions and truth-values which form the ontological scope of the system.

One of the most dramatic incidences in the history of logic is the catastrophe which overtook the Grundgesetze. When Frege was nearing the printing of the second volume in June 1902, a letter was sent to him by the young Bertrand Russell telling him that Axiom V was inconsistent.[19] In axiom V, Frege defines extensions of two concepts as identical in situation where they both have the same objects in both concepts. Russell faulted this claim by contradictory set, hence by defining a class of all sets which are not members of their set. Thus, by this claim, he derives a contradiction out of Axiom V: having a class which is a member of itself Axiom V could be defined to be not; and having a class which is not a member of itself Axiom V could be defined to be a member of itself. This was the paradox of Russell, and its finding, exposed the fact that the axiomatic basis, on which Frege had built his logicist structure on, was a fatal flaw. Upon receiving Russell's letter in this

regard, Frege admitted the contradiction with admirable intellectual honesty in the second volume of the Grundggesetze.[20]

The blow that Russell gave to the logicism of Frege was disastrous. Dummett has held that finding the inconsistency did not just indicate the presence of a technical weakness in one axiom in particular but dealt a major blow to the very conception of logic being discussed by Frege, which destroyed the naive theory of extensions using which he had built everything..[21] The paradox seems to have left Frege himself deeply uprooted, and in his later work he seems to have essentially forsaken the programme of logicism, and discovered the prospect of arithmetic having geometrical foundations after all. Whether the debris of the Grundgesetze was complete or there was anything of philosophical interest to be extracted out of the original vision of Frege is the question which has remained on the minds of philosophers ever since.

NEO-FREGEAN REVIVAL IN THE WORKS OF HUME, WRIGHT AND HALE

No doubt that major major revival attempt to Frege's logicist programme was championed by Wright, in his work on "Frege Conception of Numbers as Objects" (1983).[22] Wright enlivens Frege's assertion in *Grundlagen* which defines co-relationship as it is in Hume's principle. Thus, the number definition of Ns can be said to be co-relational to the number of Bs in situations where both A and B could be said to have "one -to-one" correspondence. Worthy of note is that Frege argued against the claim of Hume in his *Grundlagen* for the reason that Hume's Principle did not by itself determine truth value of such identity propositions/statements that involve numbers and objects beyond those numbers referred as the Julius Caesar problem. Julius Caesar Problem was introduced by Frege which questions whether abstract objects such as numbers could be identified with non-mathematical objects such as "could the number 3 mean the same thing as Julius Caesar? Frege rather

adopted definition of numbers explicitly as extensions of concepts which requires correspondence to Axiom V. Wright's neo-Fregean strategy was to reverse this decision: to take Hume's Principle, rather than Axiom V, as the foundational axiom for the derivation of arithmetic, thereby avoiding the inconsistency that had destroyed Frege's original system.[23]

This technical outcome of the neo-Fregean programme is what has been called the Frege Theorem.[24] This is the derivability of Peano axioms which is also referred as Dedekind-Peano axioms or Peano postulates. This system was developed by an Italian Mathematician Giuseppe Peano as an attempt to define properties of numbers as consisting of a set N, non-logical but constant symbols such as 0 and unary function such as S. This can be said to be an implicit finding in the work of Frege, and the first explicitly singled out and rigorously proved by Wright and it was confirmed and expanded upon in the works of George Boolos and Richard Heck.[25] Boolos proved the derivation to work in a consistent fragment of the Fregean system that contains Hume Principle but eliminates Axiom V and Heck in great detail proved the Fregean proofs of the Grundgesetze to be reproved so that they relied upon Hume Principle instead of the inconsistent Axiom V.[26]

The neo-Fregean programme has a philosophical implication that relies on the status that can be granted to the Principle of Hume. So long as the Hume Principle remains a logical truth or analytic truth that is implicit upon our understanding of concept of number, the derivation of arithmetic out of Hume Principle into second-order logic would justify a version of the original logicist thesis of Frege; namely that arithmetic is derivable out of logic plus an analytic principle of the notion of number.[27] At some length, Wright and Hale have argued that Hume was correct in the claim that the Principle of abstraction of numbers and its derivation by means of

equivalence relation, that is by one to one correspondence could be used as legitimate sources of mathematical knowledge.[28]

CRITICAL RESPONSES TO NEO-FREGEANISM

The neo-Fregean programme has not only received an elaborate and thriving critical literature. Although George Boolos accepted that Frege was technically right, he was profoundly skeptical that Hume Principle was a logical or analytic truth. Boolos emphasized that the Principle of Hume makes very strong existential claims, and that no principle with such dramatic ontological implications can coherently be accepted as the law of logic and logic is commonly believed to be ontologically non-committal and topic-neutral.[29] This objection questions the very foundation of neo-Fregean mathematical logic to the extent that Hume in not making Hume Principle analytic or logical, in giving arithmetic the derivation it has, the derivation is valid, but it does not prove anything about the nature of arithmetic as a type of logic.

A related criticism has been made by John Burgess in his work Fixing Frege, where he critically analyses a number of methods to fix the original system by Frege and evaluates their philosophical equivalent.[30] Burgess has defended that the neo-Fregean programme is in an impasse between adopting a conception of logic liberated enough to accommodate the Principle of Hume whereupon it risks trivialisation of the logicist thesis by inflating the nature of logic in a way that harmfully disregards the Principle or it is more austere in its conception of logic whereupon Hume Principle must be acknowledged to be a substantive mathematical axiom rather than a logical truth and in that case the logicist thesis does not succeed in its ambitious form..[31]

In a more conciliatory judgement, Øystein Linnebo has recently given an evaluation of thin objects and abstractionist epistemology..[32] Linnebo has proposed that the principles of abstraction like the Principle of Hume can be conceived as

introducing thin objects; objects whose existence is metaphysically lightweight and the identity conditions of which are wholly defined by the equivalence relations constituting them. According to this, the existential commitments of the Principle of Hume are then not ontologically debilitating in either of the senses that Boolos and Burgess presuppose since what is introduced by abstraction are not robust and efficacious entities in causal sense, but rather the referents of singular terms the reference of which are imposed by the very principle of abstraction itself..[33] It is not yet reasonable to claim how much of such thinning of ontological commitment is necessary to salvage the epistemological ambitions of logicism, a question on which philosophical polemics is still active.

ENDURING LEGACY IN FREGE'S LOGICISM

However, the final judgement in regard to whether logicism is viable philosophy of mathematics or not, there is no doubt in regard to the intellectual heritage of the Frege programme. The work of Frege sparked the school of analytic philosophy, which made the study of language and logic very central to philosophical study. His modern predicate logic protocol changed the very science of logic, substituting the Aristotelian syllogistic with a protocol of superior power and generality that was formal. The theoretical discussions of his numbers, concept, object, sense and reference still characterize the main issues that are central to philosophy of language and mathematics.[34] According to Frege, Tyler Burge has maintained that such work has made Frege an indispensable part of our knowledge of the attaching of thought and truth, making in him a frame of reference in which the objectivity of logic and mathematics might be formulated and argued with an unprecedented level of clarity.[35]

Joan Weiner has also contended that the logicist programme of Frege, even its failure had achieved a lasting philosophical significance: that it showed the richness and intricacy of the

question to which it sought the answer.[36] The epistemology of arithmetic was something that one could argue out in more or less casual terms before Frege. Thus, epistemology of arithmetic confronted the samantics of mathematical language, formal logic, theory of definition, metaphysics of abstract objects to a higher degree of rigour than hitherto attempted. The Grundlagen and the Grundgesetze outlined what a philosophical explanation of mathematics should do such that any philosophy of mathematics since then had been greatly influenced by the idea of logicism as raised by Frege.

William Demopoulos has noted that logicism as pioneered by Frege has also the effect of initiating a very distinctive form of philosophical argument, which is both technically precise and philosophically thoughtful in nature, and which refused to have technical sophistication serve in place of philosophical knowledge..[3738] This mingling of formal and philosophical merit continues to be the best thing in the analytic tradition, and it is a direct bequest on the work that Frege did to set make modern logic on the footing of philosophy of mathematics.

CONCLUSION

The logicism in Frege appraisal demonstrates a philosophical programme of unparalleled ambition, rigour and consequence. The thesis of Frege, based on the revolutionary action of the formal apparatus of his Begriffsschrift and the broad philosophical analyses of his Grundlagen, which was mathematics a branch of logic, was a critical threat to the Kantian and empiricist philosophies of mathematical knowledge that had dominated the philosophy of mathematics since more than a hundred years. The logicist programme was formally realised in the Grundgesetze, a heroic intellectual achievement, although it was ultimately ruined by the inconsistency of Axiom V to be found concurrently by Russell.

The neo-Fregean revival, whose key figures are the Principle of Hume and the Theorem of Frege, has shown that much of the original derivation of Frege can be reconstructed in a consistent framework. However, Hume's Principle either by analytic, logical or mere productive axiom, is still the subject of intense controversy. The philosophical interest of the debate has been made clear by the critical responses of Boolos, Burgess, and the constructive suggestions of Linnebo and others have given new lines of inquiry the question of the metaphysics and epistemology regarding abstract objects.

Frege's logicist programme was a transformative influence on the further history of philosophy, logic, and mathematics is unquestioned. No doubt, in modern logic, Frge has developed a corpus, with which we can still not dispense ourselves especially in the creation of a concept of number which can now be formulated with incomparable precision, matters of sense and reference and formal methods. The ineffectiveness of logicism in its original form does not in any lessen the grandeur of the accomplishment of Frege; but it only serves to witness the profundity of the problems to which he was the pioneer to apply to the full armament of modern logic.

ENDNOTES

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